COMBINATORIAL IDENTITIES AND INVERSE BINOMIAL COEFFICIENTS

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Abstract

In this note we present a method for obtaining a wide class of combinatorial identities. We give several examples, some of them have already been considered previously, and other are new.

1. Introduction

In 1981, Rockett [R, Th. 1] (see also [Pl]) proved the following. For any nonnegative integer n

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}. \tag{1}$$

In 1999, Trif [T] proved the above result using the Beta function. The present paper can be regarded as a far-reaching generalization of the ideas presented in [T]. Our main result, in its simplest form, can be stated as follows.

Theorem 1.1. *Let r, $n \geq k$ be any nonnegative integer numbers, let $f(n,k)$ be given by*

$$f(n,k) = \frac{(n+r)!}{n!} \int_{u_1}^{u_2} p^k(t) q^{n-k}(t) dt,$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1, u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x)$, $B(x)$ be the corresponding ordinary generating functions. Then

$$\sum_{n\geq 0} \left[\sum_{k=0}^{n} f(n,k) a_k b_{n-k} \right] x^n = \frac{d^r}{dx^r} \left[x^r \int_{u_1}^{u_2} A(xp(t)) B(xq(t)) dt \right].$$

As an easy consequence of Theorem 1.1 we get a family of identities, including the one presented above.

Example 1.2. (see [JS]) *Let $a_n = a^n$ and $b_n = b^n$ for all $n \geq 0$, and let $a+b \neq 0$. So the corresponding generating functions are $A(x) = (1-ax)^{-1}$ and $B(x) = (1-bx)^{-1}$.*

It is easy to see that

$$\binom{s}{r}^{-1} = (s+1) \int_0^1 t^r (1-t)^{s-r} dt, \tag{2}$$

for all nonnegative real numbers s and r such that $s \geq r$.

By Theorem 1.1 and equation (2),

$$\begin{aligned}\sum_{n\geq 0} x^n \sum_{k=0}^{n} a^k b^{n-k}\binom{n}{k}^{-1} &= \frac{d}{dx}\left(x\int_0^1 \frac{1}{(1-axt)(1-bx+bxt)}dt\right)\\ &= \frac{d}{dx}\left(\frac{-\ln(1-ax)-\ln(1-bx)}{a+b-abx}\right),\end{aligned}$$

and after simple transformations we get

$$\sum_{k=0}^{n} a^k b^{n-k}\binom{n}{k}^{-1} = \frac{n+1}{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)^{n+1}}\sum_{k=1}^{n+1}\frac{\left(a^k+b^k\right)\left(\frac{1}{a}+\frac{1}{b}\right)^k}{k}$$

for any nonnegative integer n. In particular, for $a=b=1$, we get (1).

Example 1.3. *Let us define $a_n = n$, $b_n = 1$ for $n \geq 0$. By Theorem 1.1 and equation (2) it is easy to see that*

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n} k\binom{n}{k}^{-1}\right]x^n = \frac{-2x\ln(1-x)}{(2-x)^3} - \frac{x(3x-4)}{(2-x)^2(1-x)^2}.$$

Hence, for any nonnegative integer n

$$\sum_{k=0}^{n} k\binom{n}{k}^{-1} = \frac{1}{2^n}\left[(n+1)(2^n-1)+\sum_{k=0}^{n-2}\frac{(n-k)(n-k-1)2^{k-1}}{k+1}\right].$$

In the rest of the paper, we prove Theorem 1.1 and generalize it to functions represented by integrals over a real d-dimensional domain. We present several examples; some of them have been considered previously, and other are new. For combinatorial identities yields from integral represtation in the complex domain see [E].

2. One-dimensional case

First of all, let us prove Theorem 1.1. Let $f(n,k)$ be as in the statement of the theorem. Then

$$\sum_{k=0}^{n} f(n,k)a_n b_{n-k} = \frac{(n+r)!}{n!}\int_{u_1}^{u_2}\sum_{k=0}^{n} a_k p^k(t) b_{n-k} q^{n-k}(t)dt,$$

which means that

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} f(n,k)a_n b_{n-k} = \sum_{n\geq 0}\left[\frac{(n+r)!x^n}{n!}\int_{u_1}^{u_2}\sum_{k=0}^{n} a_k p^k(t) b_{n-k} q^{n-k}(t)\right]dt.$$

Let $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$; hence

$$\sum_{n\geq 0}\sum_{k=0}^{n} f(n,k)a_k b_{n-k}x^n = \frac{d^r}{dx^r}\left[x^r\int_{u_1}^{u_2} A(xp(t))B(xq(t))dt\right],$$

which means that Theorem 1.1 holds. □

Now, we present other applications of Theorem 1.1.

Example 2.1. *Immediately, by equation (2) and Theorem 1.1, we get for any nonnegative integer numbers c and d*

$$\sum_{n\geq 0} x^{cn} \sum_{k=0}^{n} \binom{cn}{dk}^{-1} = \frac{d}{dx}\int_0^1 \frac{x\cdot dt}{(1-(1-t)^c x^c)(1-t^d(1-t)^{c-d}x^c)}.$$

For $c=d=2$, it is easy to get for any nonnegative integer n

$$\sum_{k=0}^{n}\binom{2n}{2k}^{-1} = \frac{n(2n+1)}{2^{2n+2}}\sum_{k=0}^{2n+1}\frac{2^k}{k+1}.$$

Theorem 2.2. *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences, $A(x)$ and $B(x)$ be the corresponding ordinary generating functions and μ be the differential operator of the first order defined by $\mu(f) = \frac{d}{dx}(x\cdot f)$. Then, for any positive integer m*

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}^{-m} a_k b_{n-k}\right] x^n =$$

$$\mu^m\left[\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{m\ times} A(xt_1t_2\dots t_m)B((1-t_1)(1-t_2)\cdots(1-t_m)x)dt_1dt_2\cdots dt_m\right].$$

Proof. Using equation (2) we get

$$\binom{n}{k}^{-m} = (n+1)^m\left[\int_0^1 t^k(1-t)^{n-k}dt\right]^m,$$

which means that

$$\binom{n}{k}^{-m} = (n+1)^m \underbrace{\int_0^1\cdots\int_0^1}_{m\ \text{times}} (t_1t_2\dots t_m)^k((1-t_1)(1-t_2)\dots(1-t_m))^{n-k}dt_1\cdots dt_m.$$

So similarly to proof of Theorem 1.1, this theorem holds. □

Now let us find another representation for $\binom{n}{k}^{-m}$.

Proposition 2.3. *For any nonnegative integers n, m*

$$\sum_{k=0}^{n}\binom{n}{k}^{-m} = (n+1)^m\sum_{k=0}^{n}\left[\sum_{i=0}^{k}\frac{(-1)^i}{n-k+1+i}\binom{k}{i}\right]^m.$$

Proof. By Equation (2) we get for all positive integer m

$$\binom{n}{k}^{-m} = (n+1)^m\left(\int_0^1 t^k(1-t)^{n-k}dt\right)^m,$$

which means that

$$\binom{n}{k}^{-m} = (n+1)^m\left[\int_0^1\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}t^{k+i}dt\right]^m,$$

hence the proposition holds. □

The above proposition and equation (1) yield the following.

Corollary 2.4. *For any nonnegative integer n,*

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}=(n+1)\sum_{k=0}^{n}\frac{1}{(n+1-k)2^k}=(n+1)\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(-1)^j}{n-k+1+j}\binom{k}{j}.$$

Corollary 2.5. *For any nonnegative integer number n,*

$$\begin{aligned}\sum_{k=0}^{n}\tbinom{n}{k}^{-2} &= (n+1)^2\sum_{k=0}^{n}\left[\sum_{i=0}^{k}\tfrac{(-1)^i}{n-k+1+i}\tbinom{k}{i}\right]^2=\\ &=(n+1)^2\sum_{k=0}^{n}\tfrac{2}{n-k+1}\sum_{j=0}^{k}\tfrac{(-1)^j}{n+2+i}\tbinom{k}{i}.\end{aligned}$$

Proof. By Proposition 2.3 the first equality holds. Now let us prove the second equality. By Theorem 2.2 we get

$$\sum_{n\geq 0}x^n\sum_{k=0}^{n}\binom{n}{k}^{-2}=\mu^2\left[\int_0^1\int_0^1\frac{1}{(1-tux)(1-(1-t)(1-u)x)}dudt\right],$$

therefore

$$\sum_{n\geq 0}x^n\sum_{k=0}^{n}\binom{n}{k}^{-2}=\mu^2\left[\int_0^1\frac{-2\ln(1-tx)}{x(1-t(1-t)x)}dt\right].$$

Hence, since $\ln(1-tx)=\sum_{n\geq 1}\frac{-t^nx^n}{n}$ and $\frac{1}{1-t(1-t)x}=\sum_{n\geq 0}t^n(1-t)^nx^n$, the second equality holds. $\square$

3. Generalization: d-dimensional case

The following result, which is a generalization of Theorem 1.1, gives us a general method for obtaining combinatorial identities.

Theorem 3.1. *Let X be a multiset of variables x_j, where $j=1,2,\dots,d+1$, and let $X'=\{x_{i_1},\dots,x_{i_l}\}$ be the underlying set. Let $g(t)$ and $f_j(t)$, $j=1,2,\dots,d$ be any $d+1$ functions such that $\phi(x_{i_1},\dots,x_{i_l})=g(x_{d+1})\prod_{j=1}^{d}f_j(x_j)$ is a function defined on a l-dimensional domain D. Let r be a nonnegative integer number, and let $f(k_1,k_2,\dots,k_d)$ be given by*

$$f(k_1,k_2,\dots,k_d)=\frac{(k_1+\cdots+k_d+r)!}{(k_1+\cdots+k_d)!}\int_D\phi(x_{i_1},\dots,x_{i_l})\,dx_{i_1}\dots dx_{i_l}.$$

Then for any sequences $\{a_n^{(j)}\}_{n\geq 0}$, $j=1,2,\dots d$,

$$\begin{aligned}&\sum_{n\geq 0}\sum_{k_1+\cdots+k_d=n}f(k_1,k_2,\dots,k_d)x^n\prod_{j=1}^{d}a_{k_i}^{(j)}=\\ &\tfrac{d^r}{dx^r}\left[x^r\int_D g(x_{d+1})\prod_{j=1}^{d}A_j(xf_j(x_j))dx_{i_1}\dots dx_{i_l}\right],\end{aligned}$$

where $A_j(x)$ is the ordinary generating function of the sequence $\{a_n^{(j)}\}_{n\geq 0}$.

Another way to generalize Theorem 1.1 is the following. Let V be the hyperplane defined by $\sum_{i=1}^{d}\left(\frac{x_i}{a_i}\right)^{p_i}=1$ where $x_i\geq 0$ for all $i=1,2,\dots d$. If $p_i\geq 0$ for all i, then the *Dirichlet's integral* is defined by

$$\int_V \prod_{j=1}^{d} x_j^{\alpha_j-1} dx_1 \cdots dx_d = \frac{a_1^{\alpha_1}\cdots a_d^{\alpha_d}}{p_1\cdots p_d} \frac{\Gamma\left(\frac{\alpha_1}{p_1}\right)\cdots\Gamma\left(\frac{\alpha_d}{p_d}\right)}{\Gamma\left(1+\frac{\alpha_1}{p_1}+\cdots+\frac{\alpha_d}{p_d}\right)}. \tag{3}$$

So for $p_j = 1$, $a_j = 1$, and $\sum_{j=1}^{d} \alpha_j = n$ we obtain

$$\binom{n}{\alpha_1, \cdots, \alpha_d}^{-m} = \frac{(n+d-1)!^m}{n!^m} \left(\int_{x_1+\cdots+x_d=1} x_1^{\alpha_1} \cdots x_d^{\alpha_d} dx_1 \cdots dx_d \right)^m . \tag{4}$$

Hence, Theorem 3.1, Theorem 1.1 and equation (3) yield the following.

Theorem 3.2. *Let* $\{a_n^{(j)}\}_{n\geq 0}$ *be any sequence for all* $j = 1, 2, \ldots, d$, *and let* ν *be the differential operator of the* $(d-1)$*th order defined by* $\nu_d(f) = \frac{d^{d-1}}{dx^{d-1}}(x^{d-1}f)$. *Then*

$$\sum_{n\geq 0} x^n \sum_{\alpha_1+\cdots+\alpha_d=n} \binom{n}{\alpha_1,\cdots,\alpha_d}^{-m} \prod_{j=1}^{d} a_{\alpha_j}^{(j)} =$$

$$= \nu_d^m \left[\underbrace{\int_V \cdots \int_V}_{m \text{ times}} \prod_{j=1}^{d} A_j(x x_{j,1} x_{j,2} \cdots x_{j,m}) \prod_{i=1,j=1}^{d,m} dx_{i,j} \right],$$

where V *is the hyperplane defined by* $x_1 + x_2 + \cdots + x_d = 1$, $A_j(x)$ *is the ordinary generating function of sequence* $\{a_n^{(j)}\}_{n\geq 0}$, $j = 1, 2, \ldots, d$.

Example 3.3. *(see Carlson* [C, Chapter 8]*) Let* $a_n^{(j)} = \binom{2n}{n}$ *for* $n \geq 0$, $j = 1, 2, \ldots, d$, *and* $m = 1$. *By Theorem 3.2 and equation (4) it is easy to see that*

$$\sum_{n\geq 0} x^n \sum_{\alpha_1+\cdots+\alpha_d=n} \binom{n}{\alpha_1,\cdots,\alpha_d}^{-1} \prod_{j=1}^{d} \binom{2\alpha_j}{\alpha_j} =$$

$$= \frac{d^{d-1}}{dx^{d-1}} x^{d-1} \left[\int_{x_1+\cdots+x_d=1} \prod_{j=1}^{d} \frac{1}{\sqrt{1-4xx_j}} \prod_{j=1}^{d} dx_j \right],$$

As numerical example, for $d = 2$, *equating the coefficients at* x^n, *we get*

$$\sum_{j=0}^{n} \binom{n}{j}^{-1} \binom{2j}{j} \binom{2n-2j}{n-j} = \sum_{j=0}^{n} 2^{n-j} \binom{2j}{j}.$$

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